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Propagating modes of a non-Abelian tensor gauge field of second rank

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Abstract
In the non-Abelian tensor gauge field theory a lower-rank field is represented by a general nonsymmetric tensor and describes the propagation of charged bosons of helicities two and zero. We clarify and prove this result from different perspectives which would include generalized Bianchi identities and the analysis of the corresponding partial differential equation. We suggest a new method for counting propagating modes in general gauge field theories. We derive also the expression for the energy–momentum tensor and confirm that its nonzero components get contribution only from helicity-two and helicity-zero states. We extended this analysis considering the interaction between two currents caused by the exchange of a tensor gauge field and proved that the residue at the pole is the sum of three terms each of which describes positive norm polarizations of helicity-two and helicity-zero bosons.

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1. Introduction

An infinite tower of massive particles of high spin naturally appears in the spectrum of different string field theories [1–7]. It is generally expected that in the tensionless limit or, what is equivalent, at high-energy and fixed-angle scattering [8–10] the string spectrum becomes effectively massless and it is of great importance to find out the corresponding Lagrangian and its genuine symmetries [11–18].

In quantum field theory, the Lagrangian of *free massless* Abelian tensor gauge fields has been formulated in [19–29]. The problem of introducing *interactions* appears to be much more complex and there has been important progress in defining self-interaction of higher-spin fields [30–41].

A possible extension of the gauge principle which defines the interaction of non-Abelian tensor gauge fields has been made recently in [42]. Recall that non-Abelian gauge fields are defined as rank-$(s+1)$ tensor gauge fields $A^a_{\mu\lambda_1\cdots\lambda_s}$ [1] and that one can construct infinite series of forms $\mathcal{L}_s (s = 1, 2, \ldots)$ and $\mathcal{L}'_s (s = 2, 3, \ldots)$ which are invariant with respect to the *extended gauge transformations* [42]. These forms are quadratic in the field strength tensors $G^a_{\mu\nu,\lambda_1\cdots\lambda_s}$. The resulting gauge-invariant Lagrangian defines cubic and quartic self-interactions of charged gauge quanta carrying a spin larger than one [42–46]. The gauge-invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form [42–44]

$$\mathcal{L} = \mathcal{L}_1 + g_2\mathcal{L}_2 + g'_2\mathcal{L}'_2 + \cdots, \tag{1.1}$$

where $\mathcal{L}_1$ is the Yang–Mills Lagrangian. For the lower-rank tensor gauge fields the Lagrangian has the following form [42–44]:

$$\begin{aligned}
\mathcal{L}_1 &= -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},\\
\mathcal{L}_2 &= -\tfrac{1}{4}G^a_{\mu\nu,\lambda}G^a_{\mu\nu,\lambda} - \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu,\lambda\lambda},\\
\mathcal{L}'_2 &= +\tfrac{1}{4}G^a_{\mu\nu,\lambda}G^a_{\mu\lambda,\nu} + \tfrac{1}{4}G^a_{\mu\nu,\nu}G^a_{\mu\lambda,\lambda} + \tfrac{1}{2}G^a_{\mu\nu}G^a_{\mu\lambda,\nu\lambda},
\end{aligned} \tag{1.2}$$

where the generalized field strength tensors are

$$\begin{aligned}
G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,\\
G^a_{\mu\nu,\lambda} &= \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} + g f^{abc}\left(A^b_\mu A^c_{\nu\lambda} + A^b_{\mu\lambda}A^c_\nu\right),\\
G^a_{\mu\nu,\lambda\rho} &= \partial_\mu A^a_{\nu\lambda\rho} - \partial_\nu A^a_{\mu\lambda\rho} + g f^{abc}\left(A^b_\mu A^c_{\nu\lambda\rho} + A^b_{\mu\lambda}A^c_{\nu\rho} + A^b_{\mu\rho}A^c_{\nu\lambda} + A^b_{\mu\lambda\rho}A^c_\nu\right)\cdots.
\end{aligned} \tag{1.3}$$

The definition of the Lagrangian forms $\mathcal{L}_s$ and $\mathcal{L}'_s$ for higher-rank fields can be found in the previous publications [42–44]. The above expressions define interacting gauge field theory with infinite many gauge fields. Not much is known about physical properties of such gauge field theories and in the present paper we shall focus our attention on the lower-rank tensor gauge field $A^a_{\mu\lambda}$, which in this theory is a general nonsymmetric tensor with $4 \times 4 = 16$ spacetime components (or $d \times d = d^2$ in d-dimensions)[2].

Each term in the Lagrangian (1.1) is separately gauge invariant; therefore, extended gauge invariance does not fix the value of the parameters g_s and g'_s. It has been found that if $g'_2 = g_2$ in (1.1) the quadratic part of the Lagrangian $\mathcal{L}$ for the field $A^a_{\mu\lambda}$ describes the propagation of helicity-two and helicity-zero $\lambda = \pm 2, 0$ charged gauge bosons [42–44]. Otherwise if one takes $g'_2 \neq g_2$ then there will be negative norm states in the spectrum. To find out why it happens is our main concern in this paper.

In the following two sections we analyze this question from different perspectives which would include generalized Bianchi identities for the field strength tensor $G^a_{\mu\nu,\lambda_1\cdots\lambda_s}$ and the direct analysis of the partial differential equation which describes in this theory the propagation of the tensor gauge field $A^a_{\mu\lambda}$. For that we suggested a new method for counting propagating modes in general gauge field theories in the form of equation (3.7) and shall apply it to the tensor gauge field theory. This clarifies and proves that the second-rank tensor gauge field $A^a_{\mu\lambda}$ describes three polarizations $\lambda = \pm 2, 0$ only when $g'_2 = g_2$.

In section 4, we derive the expression for the energy–momentum tensor (4.11), (4.16) and confirm that its nonzero components get contribution only from helicity-two and helicity-zero states.

In section 5, we extend this analysis to the interaction of two tensor currents caused by the exchange of tensor gauge bosons and prove that the residue at the pole is the sum of three terms each of which describes positive norm polarizations of helicities $\lambda = \pm 2, 0$.

[1] Tensor gauge fields $A^a_{\mu\lambda_1\cdots\lambda_s}(x)$, $s = 0, 1, 2, \ldots$ are totally symmetric with respect to the indices $\lambda_1 \cdots \lambda_s$. *A priori* the tensor fields have no symmetries with respect to the first index μ. In particular we have $A^a_{\mu\lambda} \neq A^a_{\lambda\mu}$ and $A^a_{\mu\lambda\rho} = A^a_{\mu\rho\lambda} \neq A^a_{\lambda\mu\rho}$. The adjoint group index $a = 1, \ldots, N^2 - 1$ in the case of $SU(N)$ gauge group.

[2] One should multiply these numbers by the dimension of the gauge group, $N^2 - 1$ in the case of $SU(N)$.

2. Gauge symmetries and current conservation

The Lagrangian (1.1), (1.2) can be represented as a sum of two terms,

$$\mathcal{L} = \mathcal{K} + \mathcal{L}_{\text{int}},$$

where the first term is quadratic in fields and the second one defines the cubic and quartic interaction of fields. To analyze the particle spectrum of the theory we have to consider the kinetic term $\mathcal{K}$; therefore, we shall take the coupling constant g equal to zero in (1.3). As it follows from (1.2) the kinetic term describing the propagation of the tensor gauge field $A^a_{\mu\lambda} \neq A^a_{\lambda\mu}$ has the following form [42–44]:

$$\mathcal{K} = g_2\left(-\tfrac{1}{4}F^a_{\mu\nu,\lambda}F^a_{\mu\nu,\lambda}\right) + g_2'\left(\tfrac{1}{4}F^a_{\mu\nu,\lambda}F^a_{\mu\lambda,\nu} + \tfrac{1}{4}F^a_{\mu\nu,\nu}F^a_{\mu\lambda,\lambda}\right), \tag{2.1}$$

where

$$F^a_{\mu\nu,\lambda} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}. \tag{2.2}$$

It is invariant with respect to the gauge transformation δ,

$$\delta A^a_{\mu\lambda} = \partial_\mu \xi^a_\lambda, \tag{2.3}$$

because $\delta_\xi F^a_{\mu\nu,\lambda} = 0$. When $g_2 = g_2'$, $\mathcal{K}$ becomes invariant also with respect to the *complementary* gauge transformation $\tilde{\delta}$ [44],

$$\tilde{\delta} A^a_{\mu\lambda} = \partial_\lambda \eta^a_\mu. \tag{2.4}$$

The field strength tensor $F^a_{\mu\nu,\lambda}$ (2.2) transforms with respect to this transformation as follows:

$$\tilde{\delta}_\eta F^a_{\mu\nu,\lambda} = \partial_\lambda\left(\partial_\mu \eta^a_\nu - \partial_\nu \eta^a_\mu\right). \tag{2.5}$$

The kinetic term $\mathcal{K}$ is obviously invariant with respect to the first group of gauge transformations $\delta\mathcal{K} = 0$, but it is less trivial to see that it is also invariant with respect to the complementary gauge transformations $\tilde{\delta}$. The $\tilde{\delta}$ transformation of $\mathcal{K}$ is

$$\begin{aligned}\tilde{\delta}\mathcal{K} &= -\tfrac{1}{2}F^a_{\mu\nu,\lambda}\partial_\lambda\left(\partial_\mu\eta^a_\nu - \partial_\nu\eta^a_\mu\right) + \tfrac{1}{2}F^a_{\mu\nu,\lambda}\partial_\nu\left(\partial_\mu\eta^a_\lambda - \partial_\lambda\eta^a_\mu\right) + \tfrac{1}{2}F^a_{\mu\nu,\nu}\partial_\lambda\left(\partial_\mu\eta^a_\lambda - \partial_\lambda\eta^a_\mu\right)\\ &= \tfrac{1}{2}F^a_{\mu\nu,\lambda}\partial_\lambda\partial_\nu\eta^a_\mu + \tfrac{1}{2}F^a_{\mu\nu,\nu}\partial_\lambda\left(\partial_\mu\eta^a_\lambda - \partial_\lambda\eta^a_\mu\right),\end{aligned}$$

where we combined the first, the second and the fourth terms and used the fact that the third term is identically equal to zero. Just from the symmetry properties of the field strength tensor it is not obvious why the rest of the terms are equal to zero. Therefore we shall use the explicit form of the field strength tensor $F^a_{\mu\nu,\lambda}$, which gives

$$\tilde{\delta}\mathcal{K} = \tfrac{1}{2}\left(\partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}\right)\partial_\lambda\partial_\nu\eta^a_\mu + \tfrac{1}{2}\left(\partial_\mu A^a_{\nu\nu} - \partial_\nu A^a_{\mu\nu}\right)\partial_\lambda\left(\partial_\mu\eta^a_\lambda - \partial_\lambda\eta^a_\mu\right).$$

From the corresponding action $S_0 = \int \mathcal{K}\,\mathrm{d}x$, after partial differentiation we shall get that the term $\partial_\mu A^a_{\nu\nu} \cdot \partial_\lambda\left(\partial_\mu\eta^a_\lambda - \partial_\lambda\eta^a_\mu\right)$ gives a zero contribution and the rest of the terms cancel each other $\int\left(\tfrac{1}{2}(\partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda})\cdot\partial_\lambda\partial_\nu\eta^a_\mu - \tfrac{1}{2}\partial_\nu A^a_{\mu\nu}\cdot\partial_\lambda\left(\partial_\mu\eta^a_\lambda - \partial_\lambda\eta^a_\mu\right)\right)\mathrm{d}x = 0$. This demonstrates the invariance of $\mathcal{K}$ with respect to δ and $\tilde{\delta}$ transformations when $g_2 = g_2'$ in (1.1) [42].

Let us now consider the *interaction* of the tensor gauge field $A^a_{\mu\lambda}$ defined by the total Lagrangian (1.1), (1.2). In order to see what type of restrictions are imposed on the interaction we shall consider the full equation of motion when $g \neq 0$. It follows from the Lagrangian (1.1) that

$$\partial_\mu F^a_{\mu\nu,\lambda} - \tfrac{1}{2}\left(\partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho}\right) = J^a_{\nu\lambda}(g, A), \tag{2.6}$$

where all terms containing a coupling constant g are written on the lhs, see [42–44] for details. This equation contains two terms $\partial_\mu F^a_{\mu\nu,\lambda}$ and $-\tfrac{1}{2}\left(\partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho}\right)$, which arise from $\mathcal{L}_2$ and $\mathcal{L}_2'$ respectively. The derivatives over ∂_ν of both terms in the equation are equal to zero separately. Indeed, due to the antisymmetric properties of the field strength tensor $F^a_{\mu\nu,\lambda}$ under the exchange of μ and ν we have

$$\partial_\nu \partial_\mu F^a_{\mu\nu,\lambda} = 0,$$

as well as

$$-\tfrac{1}{2}\partial_\nu \left\{ \partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho} \right\} = 0.$$

Thus it follows from (2.6) that

$$\partial_\nu J^a_{\nu\lambda} = 0. \tag{2.7}$$

Hence, the current $J^a_{\nu\lambda}$ must be divergenceless over its first index. Now, let us take derivative over ∂_λ of the lhs of equation (2.6), that is, the derivative over the second index of the nonsymmetric current $J^a_{\nu\lambda}$. We see that

$$\partial_\lambda \partial_\mu F^a_{\mu\nu,\lambda} \neq 0,$$

as well as

$$\begin{aligned}&-\tfrac{1}{2}\partial_\lambda \left\{ \partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho} \right\} \\ &\qquad = -\tfrac{1}{2}\partial_\lambda \left\{ \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho} \right\} \neq 0.\end{aligned}$$

Thus, it is not obvious to see the conservation of the nonsymmetric current $J^a_{\nu\lambda}$ with respect to its second index λ. Therefore we have to use the explicit form of the field strength tensor $F^a_{\mu\nu,\lambda} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}$, this gives

$$\begin{aligned}&\partial_\lambda \partial_\mu F^a_{\mu\nu,\lambda} - \tfrac{1}{2}\partial_\lambda \left\{ \partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho} \right\} \\ &\qquad = \partial_\lambda \partial_\mu F^a_{\mu\nu,\lambda} - \tfrac{1}{2}\partial_\lambda\partial_\mu F^a_{\lambda\nu,\mu} - \tfrac{1}{2}\partial^2 F^a_{\mu\nu,\mu} - \tfrac{1}{2}\partial_\nu\partial_\mu F^a_{\mu\rho,\rho} \\ &\qquad = \tfrac{1}{2}\partial_\lambda\partial_\mu\left(\partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}\right) - \tfrac{1}{2}\partial^2\left(\partial_\mu A^a_{\nu\mu} - \partial_\nu A^a_{\mu\mu}\right) \\ &\qquad\quad -\tfrac{1}{2}\partial_\nu\partial_\mu\left(\partial_\mu A^a_{\rho\rho} - \partial_\rho A^a_{\mu\rho}\right) = 0.\end{aligned}$$

Therefore the sum of the two nonzero expressions presented above are equal to zero, thus [42–44]

$$\partial_\lambda J^a_{\nu\lambda} = 0. \tag{2.8}$$

The natural question which arises here is connected with the fact that in order to see these cancelations one should use the explicit form of the field strength tensor $F^a_{\mu\nu,\lambda}$, and it remains a mystery, why this takes place only when the relative coefficient between the invariant forms $\mathcal{L}_2$ and $\mathcal{L}'_2$ is equal to one ($g_2 = g'_2$ in (1.1)) [42].

Our main concern therefore is to understand the general reason for these cancelations without referring to the explicit form of the field strength tensor. As we shall see, the deep reason for these cancelations lies in the Bianchi identity (A.4), (A.5) for the free-field strength tensor

$$\partial_\mu F^a_{\nu\lambda,\rho} + \partial_\nu F^a_{\lambda\mu,\rho} + \partial_\lambda F^a_{\mu\nu,\rho} = 0, \tag{2.9}$$

which we shall derive in appendix A. Indeed, we can evaluate the derivative of the lhs of equation (2.6) to the following form:

$$\begin{aligned}&\partial_\lambda \left\{ \partial_\mu F^a_{\mu\nu,\lambda} - \tfrac{1}{2}\left(\partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho} \right)\right\} \\ &\qquad = -\tfrac{1}{2}\left\{ \partial^2 F^a_{\mu\nu,\mu} + \partial_\mu\partial_\nu F^a_{\mu\rho,\rho} + \partial_\mu\partial_\lambda F^a_{\nu\lambda,\mu} \right\},\end{aligned} \tag{2.10}$$

where we have used only the antisymmetric property of $F_{\mu\nu,\lambda}$ to cancel the second term and to combine the first one with the third one of the lhs of the above equation. Now, we shall take advantage of the Bianchi identity. Taking the derivative of the Bianchi identity (2.9) over ∂_μ and setting $\nu = \rho$ we get

$$\partial^2 F^a_{\mu\nu,\mu} + \partial_\mu\partial_\nu F^a_{\mu\rho,\rho} + \partial_\mu\partial_\lambda F^a_{\nu\lambda,\mu} \equiv 0 \tag{2.11}$$

and can clearly see that the last expression in (2.10) coincides with the lhs of this contracted Bianchi identity and is therefore equal to zero. Thus (2.8) holds, $\partial_\lambda J^a_{\nu\lambda} = 0$.

In other words, if one repeats these calculations for arbitrary coefficients g_2 and g_2' in the Lagrangian (1.1) $g_2\mathcal{L}_2 + g_2'\mathcal{L}_2'$, then the last expression in parentheses (2.10) will take the form

$$\partial^2 F^a_{\mu\nu,\mu} + \partial_\mu\partial_\nu F^a_{\mu\rho,\rho} + \left(2\frac{g_2}{g_2'} - 1\right)\partial_\mu\partial_\lambda F^a_{\nu\lambda,\mu}.$$

Comparing it with the Bianchi identity (2.11) one can see that it is equal to zero only if $g_2 = g_2'$ and therefore only in that case (2.8) holds.

It seems that this situation is similar to that in gravity, where both tensors $R_{\mu\nu}$ and $g_{\mu\nu}R$ have correct transformation properties and therefore can be present in the equation of motion [51]

$$R_{\mu\nu} - c g_{\mu\nu} R = T_{\mu\nu} \tag{2.12}$$

with unknown coefficient c, but the Bianci identity $R^{\mu}_{\nu;\mu} - (1/2)R_{;\nu} = 0$ tells that the coefficient c should be taken equal to $1/2$ [52].

In the following section, we shall present a general method for counting propagating modes in gauge field theories. The number of propagating modes can be expressed in the form (3.7). This method does not require gauge fixing, it simply gives a general solution of the partial differential equation.

3. Counting propagating modes

As we have seen above, the equation of motion (2.6) which describes the propagation and the interaction of the second-rank tensor gauge field ($A^a_{\mu\lambda} \neq A^a_{\lambda\mu}$) has the following form[3] [42–44]:

$$\partial^\mu F^a_{\mu\nu,\lambda} - \tfrac{1}{2}\left(\partial^\mu F^a_{\mu\lambda,\nu} + \partial^\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^{a\ \mu}_{\mu\nu,} + \eta_{\nu\lambda}\partial^\mu F^{a\rho}_{\mu\rho,}\right) = J^a_{\nu\lambda}(g, A), \tag{3.1}$$

where $F^a_{\mu\nu,\lambda} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}$. The equivalent form of the equations of motion (2.6) in terms of the gauge field is [42–44]

$$\begin{aligned}\partial^2\left(A^a_{\nu\lambda} - \tfrac{1}{2}A^a_{\lambda\nu}\right) - \partial_\nu\partial^\mu\left(A^a_{\mu\lambda} - \tfrac{1}{2}A^a_{\lambda\mu}\right) - \partial_\lambda\partial^\mu\left(A^a_{\nu\mu} - \tfrac{1}{2}A^a_{\mu\nu}\right) + \partial_\nu\partial_\lambda\left(A^{a\ \mu}_{\mu} - \tfrac{1}{2}A^{a\ \mu}_{\mu}\right)\\ + \tfrac{1}{2}\eta_{\nu\lambda}\left(\partial^\mu\partial^\rho A^a_{\mu\rho} - \partial^2 A^{a\ \mu}_{\mu}\right) = J^a_{\nu\lambda}(g, A).\end{aligned} \tag{3.2}$$

In momentum space this type of second-order partial differential equations can always be represented as matrix equation of the following general form:

$$H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k) A^a_{\gamma\acute{\gamma}} = J^a_{\alpha\acute{\alpha}}, \tag{3.3}$$

where $H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)$ is a matrix operator quadratic in momentum k_μ. In our case it has the following form [42–44]:

$$\begin{aligned}H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k) = \left(-\eta_{\alpha\gamma}\eta_{\acute{\alpha}\acute{\gamma}} + \tfrac{1}{2}\eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma} + \tfrac{1}{2}\eta_{\alpha\acute{\alpha}}\eta_{\gamma\acute{\gamma}}\right)k^2 + \eta_{\alpha\gamma}k_{\acute{\alpha}}k_{\acute{\gamma}} + \eta_{\acute{\alpha}\acute{\gamma}}k_\alpha k_\gamma\\ - \tfrac{1}{2}\left(\eta_{\alpha\acute{\gamma}}k_{\acute{\alpha}}k_\gamma + \eta_{\acute{\alpha}\gamma}k_\alpha k_{\acute{\gamma}} + \eta_{\alpha\acute{\alpha}}k_\gamma k_{\acute{\gamma}} + \eta_{\gamma\acute{\gamma}}k_\alpha k_{\acute{\alpha}}\right),\end{aligned} \tag{3.4}$$

[3] From now on the Lorentz indices of the tensor fields are raised and lowered with flat spacetime metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$.

with the property that $H_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = H_{\gamma\acute{\gamma}\alpha\acute{\alpha}}$. First of all, we shall solve the equation in the case when there are no interactions, $J^a_{\alpha\acute{\alpha}} = 0$,

$$H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k) A^a_{\gamma\acute{\gamma}} = 0. \tag{3.5}$$

The vector space of independent solutions $A_{\gamma\acute{\gamma}}$ of this system of equations crucially depends on the rank of the matrix $H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)$. If the matrix operator H has dimension $d \times d$ and its rank is rank $H = r$, then the vector space has the dimension

$$\mathcal{N} = d - r.$$

Because the matrix operator $H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)$ explicitly depends on the momentum k_μ, its rank $H = r$ also depends on momenta and therefore the number of independent solutions $\mathcal{N}$ depends on momenta

$$\mathcal{N}(k) = d - r(k). \tag{3.6}$$

Analyzing the rank H of the matrix operator H one can observe that it depends on the value of momentum square k^2_μ. When $k^2_\mu \neq 0$—off mass-shell momenta—the vector space consists of *pure gauge fields*. When $k^2_\mu = 0$—on mass-shell momenta—the vector space consists of *pure gauge fields and propagating modes*. Therefore the number of propagating modes can be calculated from the following relation:

$$\text{number of propagating modes} = \mathcal{N}(k)|_{k^2=0} - \mathcal{N}(k)|_{k^2\neq 0} = \text{rank } H|_{k^2\neq 0} - \text{rank } H|_{k^2=0}. \tag{3.7}$$

Before considering the equation of motion for the tensor gauge field (3.5), let us consider for illustration some important examples.

3.1. Vector gauge field

The kinetic term of the Lagrangian which describes the propagation of a free vector gauge field is

$$\mathcal{K} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.8}$$

and the corresponding equation of motion in the momentum space is

$$H_\alpha{}^\gamma e_\gamma = \left(-k^2 \delta^\gamma_\alpha + k_\alpha k^\gamma\right) e_\gamma = 0, \tag{3.9}$$

where $A_\mu = e_\mu \exp(\mathrm{i}kx)$. We can always choose the momentum vector in the third direction $k^\mu = (\omega, 0, 0, k)$ and the matrix operator H takes the form

$$H_\alpha{}^\gamma = \begin{pmatrix} -k^2 & 0 & 0 & -k\omega \\ 0 & \omega^2 - k^2 & 0 & 0 \\ 0 & 0 & \omega^2 - k^2 & 0 \\ k\omega & 0 & 0 & \omega^2 \end{pmatrix}.$$

If $\omega^2 - k^2 \neq 0$, the rank of the four-dimensional matrix $H_\alpha{}^\gamma$ is rank $H|_{\omega^2-k^2\neq 0} = 3$ and the number of independent solutions is $4-3 = 1$. As one can see from the relation $H_\alpha{}^\gamma(k)k_\gamma = 0$ this solution is proportional to the momentum $e_\mu = k_\mu = (-\omega, 0, 0, k)$ and is a pure gauge field. This is a consequence of the gauge invariance of the theory $e_\mu \to e_\mu + ak_\mu$. If $\omega^2 - k^2 = 0$, then the rank of the matrix drops, rank $H|_{\omega^2-k^2=0} = 1$, and the number of independent solutions increases: $4 - 1 = 3$. These three solutions of equations (3.9) are

$$e^{(gauge)}_\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad e^{(1)}_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e^{(2)}_\gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

from which the first one is a pure gauge field ($\sim k_\gamma$), while the remaining two are the physical modes, perpendicular to the direction of the momentum. The general solution at $\omega^2 - k^2 = 0$ will be a linear combination of these three eigenvectors,

$$e_\gamma = ak_\gamma + c_1 e^{(1)}_\gamma + c_2 e^{(2)}_\gamma,$$

where a, c_1, c_2 are arbitrary constants. We see that the number of propagating modes is

$$\operatorname{rank} H|_{\omega^2-k^2\neq 0} - \operatorname{rank} H|_{\omega^2-k^2=0} = 3 - 1 = 2,$$

as it should be.

3.2. Symmetric tensor gauge field

The free gravitational field is described in terms of a symmetric second-rank tensor field $h_{\mu\nu}$ and is governed by the Einstein and Pauli–Fierz equation,

$$\partial^2 h_{\nu\lambda} - \partial_\nu \partial^\mu h_{\mu\lambda} - \partial_\lambda \partial^\mu h_{\mu\nu} + \partial_\nu\partial_\lambda h^\mu_\mu + \eta_{\nu\lambda}\left(\partial^\mu\partial^\rho h_{\mu\rho} - \partial^2 h^\mu_\mu\right) = 0, \tag{3.10}$$

which is invariant with respect to the gauge transformations

$$\delta h_{\mu\lambda} = \partial_\mu \xi_\lambda + \partial_\lambda \xi_\mu \tag{3.11}$$

that respects the symmetry properties of $A_{\mu\nu}$. The corresponding matrix operator is

$$\begin{aligned} H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k) = {} & \left\{\eta_{\alpha\acute{\alpha}}\eta_{\gamma\acute{\gamma}} - \tfrac{1}{2}(\eta_{\alpha\gamma}\eta_{\acute{\alpha}\acute{\gamma}} + \eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma})\right\}k^2 - \eta_{\alpha\acute{\alpha}}k_\gamma k_{\acute{\gamma}} - \eta_{\gamma\acute{\gamma}}k_\alpha k_{\acute{\alpha}} \\ & + \tfrac{1}{2}(\eta_{\alpha\acute{\gamma}}k_\gamma k_{\acute{\alpha}} + \eta_{\acute{\alpha}\acute{\gamma}}k_\alpha k_\gamma + \eta_{\alpha\gamma}k_{\acute{\alpha}}k_{\acute{\gamma}} + \eta_{\acute{\alpha}\gamma}k_\alpha k_{\acute{\gamma}}) \end{aligned} \tag{3.12}$$

and is a 10×10 matrix in four-dimensional spacetime with the property $H_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = H_{\acute{\alpha}\alpha\gamma\acute{\gamma}} = H_{\alpha\acute{\alpha}\acute{\gamma}\gamma} = H_{\gamma\acute{\gamma}\alpha\acute{\alpha}}$ and is presented in appendix B.

If $\omega^2 - k^2 \neq 0$, the rank of the ten-dimensional matrix $H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)$ is equal to $\operatorname{rank} H|_{\omega^2-k^2\neq 0} = 6$ and the number of independent solutions is $10 - 6 = 4$. These four symmetric solutions are pure gauge tensor fields. Indeed, if again we choose the coordinate system so that $k^\gamma = (\omega, 0, 0, k)$, then one can find the following four linearly independent solutions:

$$e_{\gamma\acute{\gamma}} = \begin{pmatrix} -\omega^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & k^2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -\omega & 0 & 0\\ -\omega & 0 & 0 & k\\ 0 & 0 & 0 & 0\\ 0 & k & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & -\omega & 0\\ 0 & 0 & 0 & 0\\ -\omega & 0 & 0 & k\\ 0 & 0 & k & 0 \end{pmatrix}, \qquad \begin{pmatrix} -2\omega & 0 & 0 & k\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ k & 0 & 0 & 0 \end{pmatrix}, \tag{3.13}$$

pure gauge field solutions of the form (3.11) $e_{\gamma\acute{\gamma}} = k_\gamma \xi_{\acute{\gamma}} + k_{\acute{\gamma}}\xi_\gamma$ as one can see from the relation

$$H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)(k_\gamma\xi_{\acute{\gamma}} + k_{\acute{\gamma}}\xi_\gamma) = 0. \tag{3.14}$$

When $\omega^2 - k^2 = 0$, then the rank of the matrix $H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k)$ drops and is equal to $\operatorname{rank} H|_{\omega^2-k^2=0} = 4$. This leaves us with $10 - 4 = 6$ solutions. These are the four pure gauge solutions (3.11) and two additional symmetric solutions representing propagating modes: the helicity states of the graviton

$$e^{(1)}_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e^{(2)}_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Thus the general solution of the equation on mass shell is

$$e_{\gamma\acute{\gamma}} = \xi_\gamma k_{\acute{\gamma}} + \xi_{\acute{\gamma}} k_\gamma + c_1 e^{(1)}_{\gamma\acute{\gamma}} + c_2 e^{(2)}_{\gamma\acute{\gamma}},$$

where c_1, c_2 are arbitrary constants. We see that the number of propagating modes is

$$\text{rank } H|_{\omega^2-k^2\neq 0} - \text{rank } H|_{\omega^2-k^2=0} = 6 - 4 = 2,$$

as it should be.

3.3. Antisymmetric tensor gauge field

The antisymmetric second-rank tensor field $B_{\mu\nu}$ is governed by the equation [47–50]

$$\partial^2 B_{\nu\lambda} - \partial_\nu \partial^\mu B_{\mu\lambda} + \partial_\lambda \partial^\mu B_{\mu\nu} = 0, \tag{3.16}$$

which is invariant with respect to the gauge transformations

$$\delta B_{\mu\lambda} = \partial_\mu \eta_\lambda - \partial_\lambda \eta_\mu \tag{3.17}$$

that respects the symmetry properties of $B_{\mu\nu}$. The corresponding matrix operator is

$$\begin{aligned} H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k) = &-\tfrac{1}{2}(\eta_{\alpha\gamma}\eta_{\acute{\alpha}\acute{\gamma}} - \eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma})k^2 \\ &- \tfrac{1}{2}(\eta_{\alpha\acute{\gamma}} k_\gamma k_{\acute{\alpha}} - \eta_{\acute{\alpha}\acute{\gamma}} k_\alpha k_\gamma + \eta_{\acute{\alpha}\gamma} k_\alpha k_{\acute{\gamma}} - \eta_{\alpha\gamma} k_{\acute{\alpha}} k_{\acute{\gamma}}) \end{aligned} \tag{3.18}$$

and is 6×6 matrix in four-dimensional spacetime with the property $H_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = -H_{\acute{\alpha}\alpha\gamma\acute{\gamma}} = -H_{\alpha\acute{\alpha}\acute{\gamma}\gamma} = H_{\gamma\acute{\gamma}\alpha\acute{\alpha}}$ and is presented in appendix B.

If $\omega^2 - k^2 \neq 0$, the rank of the six-dimensional matrix $H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)$ is equal to rank $H|_{\omega^2-k^2\neq 0} = 3$ and the number of independent solutions is $6 - 3 = 3$. These three antisymmetric solutions are pure gauge fields. Indeed, in the coordinate system $k^\gamma = (\omega, 0, 0, k)$, one can find the following three solutions:

$$e_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & k \\ 0 & 0 & -k & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.19}$$

pure gauge fields of the form (3.17) $e_{\gamma\acute{\gamma}} = k_\gamma \eta_{\acute{\gamma}} - k_{\acute{\gamma}} \eta_\gamma$, as one can see from the relation

$$H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)(k_\gamma \eta_{\acute{\gamma}} - k_{\acute{\gamma}} \eta_\gamma) = 0. \tag{3.20}$$

When $\omega^2 - k^2 = 0$, then the rank of the matrix $H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k)$ drops and is equal to rank $H|_{\omega^2-k^2=0} = 2$. This leaves us with $6 - 2 = 4$ solutions. These are the three pure gauge solutions (3.17) and the antisymmetric solution representing the propagating mode: the helicity zero state

$$e^{(A)}_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.21}$$

Thus on the mass shell the general solution of the equation is

$$e_{\gamma\acute{\gamma}} = k_\gamma \eta_{\acute{\gamma}} - k_{\acute{\gamma}} \eta_\gamma + c_3 e^{(A)}_{\gamma\acute{\gamma}},$$

where c_3 is an arbitrary constant. We see that the number of propagating modes is

$$\text{rank } H|_{\omega^2-k^2\neq 0} - \text{rank } H|_{\omega^2-k^2=0} = 3 - 2 = 1.$$

After this parenthetic discussion we shall turn to the tensor gauge theory.

3.4. Non-Abelian tensor gauge field

Now we are ready to consider equation (3.5) for the tensor gauge field $A_{\mu\lambda}$ with the matrix operator (3.4) which, in four-dimensional spacetime, is a 16×16 matrix. In the reference frame, where $k^\gamma = (\omega, 0, 0, k)$, it has the form presented in appendix B.

If $\omega^2 - k^2 \neq 0$, the rank of the 16-dimensional matrix $H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)$ is equal to rank $H|_{\omega^2-k^2\neq 0} = 9$ and the number of linearly independent solutions is $16 - 9 = 7$. These seven solutions are

$$e_{\gamma\acute{\gamma}} = \begin{pmatrix} -\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k^2 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.22}$$

pure gauge tensor potentials of the form (2.3) and (2.4)

$$e_{\gamma\acute{\gamma}} = k_\gamma \xi_{\acute{\gamma}} + k_{\acute{\gamma}} \eta_\gamma \tag{3.23}$$

as one can get convinced from the relation

$$H_{\alpha\acute{\alpha}}{}^{\gamma\acute{\gamma}}(k)(k_\gamma \xi_{\acute{\gamma}} + k_{\acute{\gamma}} \eta_\gamma) = 0, \tag{3.24}$$

which follows from the gauge invariance of the action and can be checked also explicitly.

When $\omega^2 - k^2 = 0$, then the rank of the matrix $H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k)$ drops and is equal to rank $H|_{\omega^2-k^2=0} = 6$. This leaves us with $16-6 = 10$ solutions. These are seven solutions, the pure gauge potentials (3.22), (3.23) and new three solutions representing propagating modes,

$$e^{(1)}_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e^{(2)}_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e^{A}_{\gamma\acute{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

Thus the general solution of the equation on the mass shell is

$$e_{\gamma\acute{\gamma}} = \xi_{\acute{\gamma}} k_\gamma + \eta_\gamma k_{\acute{\gamma}} + c_1 e^{(1)}_{\gamma\acute{\gamma}} + c_2 e^{(2)}_{\gamma\acute{\gamma}} + c_3 e^{(A)}_{\gamma\acute{\gamma}}, \tag{3.26}$$

where c_1, c_2, c_3 are arbitrary constants. We see that the number of propagating modes is three

$$\text{rank}\, H|_{\omega^2-k^2\neq 0} - \text{rank}\, H|_{\omega^2-k^2=0} = 9 - 6 = 3.$$

These are propagating modes of helicity-two and helicity-zero $\lambda = \pm 2, 0$ charged gauge bosons [42–44]. Indeed, if we make a rotation around the z-axis,

$$\Lambda = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

we get

$$e^{(1)\prime} = \Lambda e^{(1)} \Lambda^T = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}, \qquad e^{(2)\prime} = \Lambda e^{(2)} \Lambda^T = \begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix},$$

therefore the first two solutions describe helicity $\lambda = \pm 2$ states. On the other hand, the third, antisymmetric solution remains invariant under the Lorentz transformations; therefore it describes the helicity-zero state.

4. Energy–momentum tensor

We would like to consider the contribution of the general solution (3.26) into the energy–momentum of the tensor gauge field theory. This will test from another point of view the unitarity of the theory. One can expect that only transverse propagating modes

$$e_{\gamma\acute{\gamma}} \sim c_1 e^{(1)}_{\gamma\acute{\gamma}} + c_2 e^{(2)}_{\gamma\acute{\gamma}} + c_3 e^{(A)}_{\gamma\acute{\gamma}} \tag{4.1}$$

will contribute to the energy–momentum of the gauge fields and that the longitudinal, pure gauge fields,

$$e_{\gamma\acute{\gamma}} \sim \xi_{\acute{\gamma}} k_\gamma + \eta_\gamma k_{\acute{\gamma}}, \tag{4.2}$$

will have no contribution. Let us first begin with the free theory $g = 0$. The free Lagrangian has the form (2.1)

$$\mathcal{K} = -\tfrac{1}{4} F^a_{\mu\nu,\lambda} F^a_{\mu\nu,\lambda} + \tfrac{1}{4} F^a_{\mu\nu,\lambda} F^a_{\mu\lambda,\nu} + \tfrac{1}{4} F^a_{\mu\nu,\nu} F^a_{\mu\lambda,\lambda} \tag{4.3}$$

and the equation of motion for the $A_{\mu\nu}$ field is (2.6),

$$\partial_\mu F^a_{\mu\nu,\lambda} - \tfrac{1}{2}\left(\partial_\mu F^a_{\mu\lambda,\nu} + \partial_\mu F^a_{\lambda\nu,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda}\partial_\mu F^a_{\mu\rho,\rho}\right) = 0. \tag{4.4}$$

By definition, the energy momentum tensor for the $A_{\mu\nu}$ field is

$$T_{\mu\nu} = \partial_\mu A_{\rho\sigma} \frac{\partial\mathcal{K}}{\partial(\partial_\nu A_{\rho\sigma})} - \eta_{\mu\nu}\mathcal{K}. \tag{4.5}$$

In order to calculate the term $\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_{\rho\sigma})}$ we need the expression for the derivative of the field strength tensor,

$$\frac{\partial F_{\mu\lambda,\tau}}{\partial(\partial_\nu A_{\rho\sigma})} = (\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\lambda\nu}\eta_{\rho\mu})\eta_{\sigma\tau};$$

hence it is easy to see that

$$\frac{\partial\mathcal{K}}{\partial(\partial_\nu A_{\rho\sigma})} = -F_{\nu\rho,\sigma} + \tfrac{1}{2}(F_{\nu\sigma,\rho} - F_{\rho\sigma,\nu}) + \tfrac{1}{2}(F_{\nu\lambda,\lambda}\eta_{\rho\sigma} - F_{\rho\lambda,\lambda}\eta_{\nu\sigma})$$

and finally we shall get

$$\begin{aligned} T_{\mu\nu} = {}& -\partial_\mu A_{\rho\sigma} F_{\nu\rho,\sigma} + \tfrac{1}{4}\eta_{\mu\nu} F_{\rho\sigma,\tau} F_{\rho\sigma,\tau} \\ & + \tfrac{1}{2}\partial_\mu A_{\rho\sigma}(F_{\nu\sigma,\rho} - F_{\rho\sigma,\nu}) - \tfrac{1}{4}\eta_{\mu\nu} F_{\lambda\rho,\sigma} F_{\lambda\sigma,\rho} \\ & + \tfrac{1}{2}(\partial_\mu A_{\sigma\sigma} F_{\nu\rho,\rho} - \partial_\mu A_{\rho\nu} F_{\rho\sigma,\sigma}) - \tfrac{1}{4}\eta_{\mu\nu} F_{\rho\tau,\tau} F_{\rho\sigma,\sigma}. \end{aligned} \tag{4.6}$$

With the aid of (4.4) one can compute the derivative of the energy–momentum tensor $T_{\mu\nu}$ over its second index ν and demonstrate that it is zero,

$$\partial_\nu T_{\mu\nu} = 0. \tag{4.7}$$

The energy–momentum tensor is not uniquely defined because one can add any term of the form $\partial_\rho \Psi_{\mu\nu\rho}$,

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \partial_\rho \Psi_{\mu\nu\rho},$$

where $\Psi_{\mu\nu\rho} = -\Psi_{\mu\rho\nu}$ without changing its basic property (4.7) and the total four-momentum of the system

$$P_\mu = \int T_{\mu 0}\, \mathrm{d}V. \tag{4.8}$$

We can use this freedom to express $T_{\mu\nu}$ solely in terms of the field strength tensor $F_{\mu\nu,\lambda}$. Choosing

$$\Psi_{\mu\nu\rho} = A_{\mu\sigma}F_{\nu\rho,\sigma} - \tfrac{1}{2}(A_{\mu\sigma}F_{\nu\sigma,\rho} + A_{\mu\sigma}F_{\sigma\rho,\nu} + A_{\mu\rho}F_{\nu\lambda,\lambda} + A_{\mu\nu}F_{\lambda\rho,\lambda}), \quad (4.9)$$

which fulfills the property $\Psi_{\mu\nu\rho} = -\Psi_{\mu\rho\nu}$, and using (4.4) we can get that

$$\partial_\rho\Psi_{\mu\nu\rho} = F_{\nu\rho,\sigma}\partial_\rho A_{\mu\sigma} - \tfrac{1}{2}(F_{\nu\sigma,\rho}\partial_\rho A_{\mu\sigma} + F_{\sigma\rho,\nu}\partial_\rho A_{\mu\sigma} + F_{\nu\sigma,\sigma}\partial_\rho A_{\mu\rho} + F_{\sigma\rho,\sigma}\partial_\rho A_{\mu\nu}). \quad (4.10)$$

The sum of (4.6) and (4.10) gives the final form of the expressed in terms of field strength tensors,

$$\begin{aligned} T_{\mu\nu} = &-F_{\mu\rho,\sigma}F_{\nu\rho,\sigma} + \tfrac{1}{4}\eta_{\mu\nu}F_{\rho\sigma,\tau}F_{\rho\sigma,\tau} \\ &+ \tfrac{1}{2}(F_{\mu\rho,\sigma}F_{\nu\sigma,\rho} + F_{\mu\rho,\sigma}F_{\sigma\rho,\nu}) - \tfrac{1}{4}\eta_{\mu\nu}F_{\lambda\rho,\sigma}F_{\lambda\sigma,\rho} \\ &+ \tfrac{1}{2}(F_{\mu\sigma,\sigma}F_{\nu\rho,\rho} + F_{\rho\mu,\nu}F_{\rho\lambda,\lambda}) - \tfrac{1}{4}\eta_{\mu\nu}F_{\rho\sigma,\sigma}F_{\rho\lambda,\lambda}. \end{aligned} \quad (4.11)$$

It is easy to see that the energy–momentum tensor is traceless,

$$T = T_{\mu\mu} = 0, \quad (4.12)$$

as it should be in a massless and scale invariant theory. As it is also obvious from the final expression that it is not symmetric $T_{\mu\nu} \neq T_{\nu\mu}$. This only means that it cannot be used for the calculation of the angular momentum of the fields (see paragraphs 32 and 96 of [53]).

Now we can calculate the contribution of the general solution (3.26) into the energy and momentum of the free gauge field. First of all, we can find that

$$F_{\mu\nu,\lambda} = \mathrm{i}(-k_\mu e_{\nu\lambda} + k_\nu e_{\mu\lambda}),$$

where $e_{\mu\nu}$ is a general solution (3.26)

$$e_{\mu\nu} = \xi_\nu k_\mu + \eta_\mu k_\nu + c_1 e^{(1)}_{\mu\nu} + c_2 e^{(2)}_{\mu\nu} + c_3 e^{(A)}_{\mu\nu}.$$

Using the following orthogonality relations:

$$\begin{aligned} &k_\mu k^\mu = 0, \qquad k_\mu e^{\mu\lambda} = k_\lambda e^{\mu\lambda} = 0 \\ &e^{(i)}_{\mu\lambda}e^{(j)}_{\mu\lambda} = e^{(i)}_{\mu\lambda}e^{(j)}_{\lambda\mu} = \delta^{ij}, \qquad \text{for} \quad i, j = 1, 2 \\ &e^{(A)}_{\mu\lambda}e^{(A)}_{\mu\lambda} = 1, \qquad e^{(A)}_{\mu\lambda}e^{(A)}_{\lambda\mu} = -1 \\ &e^{(A)}_{\mu\lambda}e^{(i)}_{\mu\lambda} = e^{(A)}_{\mu\lambda}e^{(i)}_{\lambda\mu} = 0, \qquad i = 1, 2, \end{aligned}$$

it is straightforward to see that

$$T_{\mu\nu} = \tfrac{1}{2}k_\mu k_\nu\left(c_1^2 + c_2^2 + 3c_3^2\right). \quad (4.13)$$

Thus we see that only the transverse propagating modes contribute to the energy–momentum of the field. As expected no pure gauge fields appear in the expression (4.13).

For completeness let us derive also the expression for the energy–momentum tensor in the interacting case when $g \neq 0$. The energy–momentum tensor for the full Lagrangian (1.1), (1.2) is defined as usual,

$$T_{\mu\nu} = \partial_\mu A^a_\lambda \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A^a_\lambda\right)} + \partial_\mu A^a_{\lambda\rho}\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A^a_{\lambda\rho}\right)} + \frac{1}{2}\partial_\mu A^a_{\lambda\rho\sigma}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A^a_{\lambda\rho\sigma}\right)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A^a_{\lambda\rho\sigma}\right)}\right) - \eta_{\mu\nu}\mathcal{L}.$$

Note the symmetrization of the second factor of the third term. One can easily derive the following relations:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\lambda)} &= -G^a_{\nu\lambda} + g_2\left\{-\frac{1}{2}G^a_{\nu\lambda,\tau\tau} + \frac{1}{2}\left(G^a_{\nu\tau,\lambda\tau} - G^a_{\lambda\tau,\nu\tau}\right)\right\}, \\ \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A^a_{\lambda\rho}\right)} &= g_2\left\{-G^a_{\nu\lambda,\rho} + \frac{1}{2}\left(G^a_{\nu\rho,\lambda} - G^a_{\lambda\rho,\nu}\right) + \frac{1}{2}\left(G^a_{\nu\sigma,\sigma}\eta_{\lambda\rho} - G^a_{\lambda,\sigma\sigma}\eta_{\nu\rho}\right)\right\}, \\ \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A^a_{\lambda\rho\sigma}\right)} &= g_2\left\{-\frac{1}{2}\eta_{\rho\sigma}G^a_{\nu\lambda} + \frac{1}{2}\left(\eta_{\lambda\sigma}G^a_{\nu\rho} - \eta_{\nu\sigma}G^a_{\lambda\rho}\right)\right\} \end{aligned}$$

in order to get

$$\begin{aligned}T_{\mu\nu} &= \partial_\mu A^a_\lambda\big\{-G^a_{\nu\lambda} + g_2\big[-\tfrac{1}{2}G^a_{\nu\lambda,\tau\tau} + \tfrac{1}{2}\big(G^a_{\nu\tau,\lambda\tau} - G^a_{\lambda\tau,\nu\tau}\big)\big]\big\}\\
&\quad + g_2\partial_\mu A^a_{\lambda\rho}\big\{-G^a_{\nu\lambda,\rho} + \tfrac{1}{2}\big(G^a_{\nu\rho,\lambda} - G^a_{\lambda\rho,\nu}\big) + \tfrac{1}{2}\big(G^a_{\nu\sigma,\sigma}\eta_{\lambda\rho} - G^a_{\lambda,\sigma\sigma}\eta_{\nu\rho}\big)\big\}\\
&\quad + g_2\tfrac{1}{2}\partial_\mu A^a_{\lambda\rho\sigma}\big\{-\eta_{\rho\sigma}G^a_{\nu\lambda} + \tfrac{1}{2}\big(\eta_{\lambda\sigma}G^a_{\nu\rho} + \eta_{\lambda\rho}G^a_{\nu\sigma} - \eta_{\nu\sigma}G^a_{\lambda\rho} - \eta_{\nu\rho}G^a_{\lambda\sigma}\big)\big\} - \eta_{\mu\nu}\mathcal{L}. \end{aligned} \quad (4.14)$$

Again we shall take advantage of the freedom we have to add terms of the form $\partial_\lambda\Psi_{\mu\nu\lambda}$ with $\Psi_{\mu\nu\lambda} = -\Psi_{\mu\lambda\nu}$, and express $T_{\mu\nu}$ solely in terms of the field strength tensors $G^a_{\mu\nu}$, $G^a_{\mu\nu,\lambda}$, $G^a_{\mu\nu,\lambda\rho}$. The equations of motion direct our choice of the tensor $\Psi_{\mu\nu\lambda}$,

$$\begin{aligned}\Psi^{\mu\nu\lambda} &= A^a_\mu\big[-G^a_{\lambda\nu} + g_2\big(-\tfrac{1}{2}G^a_{\lambda\nu,\rho\rho} + \tfrac{1}{2}G^a_{\lambda\rho,\nu\rho} - \tfrac{1}{2}G^a_{\nu\rho,\lambda\rho}\big)\big]\\
&\quad + g_2A^a_{\mu\rho}\big[-G^a_{\lambda\nu,\rho} + \tfrac{1}{2}\big(G^a_{\lambda\rho,\nu} - G^a_{\nu\rho,\lambda} + \eta_{\nu\rho}G^a_{\lambda\sigma,\sigma} - \eta_{\lambda\rho}G^a_{\nu\sigma,\sigma}\big)\big]\\
&\quad + g_2A^a_{\mu\rho\sigma}\big[-\tfrac{1}{2}\eta_{\rho\sigma}G^a_{\lambda\nu} + \tfrac{1}{4}\big(\eta_{\nu\sigma}G^a_{\lambda\rho} + \eta_{\nu\rho}G^a_{\lambda\sigma} - \eta_{\lambda\rho}G^a_{\nu\sigma} - \eta_{\lambda\sigma}G^a_{\nu\rho}\big)\big],\end{aligned} \quad (4.15)$$

so that the final expression for the $T_{\mu\nu} \to T_{\mu\nu} + \partial_\lambda\Psi_{\mu\nu\lambda}$ is

$$\begin{aligned}T_{\mu\nu} &= -G^a_{\mu\lambda}G^a_{\nu\lambda} + \tfrac{1}{4}\eta_{\mu\nu}G^a_{\lambda\rho}G^a_{\lambda\rho} + g_2\big\{-G^a_{\mu\lambda,\rho}G^a_{\nu\lambda,\rho} + \tfrac{1}{4}\eta_{\mu\nu}G^a_{\lambda\rho,\sigma}G^a_{\lambda\rho,\sigma}\\
&\quad - \tfrac{1}{2}\big(G^a_{\mu\lambda}G^a_{\nu\lambda,\rho\rho} + G^a_{\mu\lambda,\rho\rho}G^a_{\nu\lambda}\big) + \tfrac{1}{4}\eta_{\mu\nu}G^a_{\lambda\rho}G^a_{\lambda\rho,\sigma,\sigma}\\
&\quad + \tfrac{1}{2}\big(G^a_{\mu\lambda,\rho}G^a_{\nu\rho,\lambda} + G^a_{\lambda\mu,\rho}G^a_{\lambda\rho,\nu}\big) - \tfrac{1}{4}\eta_{\mu\nu}G^a_{\lambda\rho,\sigma}G^a_{\lambda\sigma,\rho}\\
&\quad + \tfrac{1}{2}\big(G^a_{\mu\rho,\rho}G^a_{\nu\lambda,\lambda} + G^a_{\lambda\rho,\rho}G^a_{\lambda\mu,\nu}\big) - \tfrac{1}{4}\eta_{\mu\nu}G^a_{\lambda\rho,\rho}G^a_{\lambda\sigma,\sigma}\\
&\quad + \tfrac{1}{2}\big(G^a_{\mu\lambda}G^a_{\nu\rho,\lambda,\rho} + G^a_{\lambda\mu}G^a_{\lambda\rho,\nu\rho} + G^a_{\nu\lambda}G^a_{\mu\rho,\lambda\rho} + G^a_{\lambda\rho}G^a_{\lambda\mu,\rho\nu}\big) - \tfrac{1}{2}\eta_{\mu\nu}G^a_{\lambda\rho}G^a_{\lambda\sigma,\rho\sigma}\big\}.\end{aligned} \quad (4.16)$$

It can be easily seen that $T = T_{\mu\mu} = 0$ and it reduces to (4.11) when $g = 0$ and only the second-rank field is present.

5. Interaction of currents

The interaction amplitude between two tensor currents caused by the exchange of these tensor gauge bosons can be found from (3.1)–(3.3) and has the following form [44]:

$$J'_{\mu\lambda}\Delta^{\mu\lambda\nu\rho}J_{\nu\rho}, \quad (5.1)$$

where the propagator $\Delta^{ab}_{\mu\lambda\nu\rho}$ is

$$\Delta^{ab}_{\mu\lambda\nu\rho} = \delta^{ab}\frac{\eta_{\mu\nu}\eta_{\lambda\rho} - \frac{1}{2}\eta_{\mu\lambda}\eta_{\nu\rho}}{\omega^2 - k^2}, \quad (5.2)$$

therefore

$$J'_{\mu\lambda}\Delta^{\mu\lambda\nu\rho}J_{\nu\rho} = J'_{\mu\lambda}\frac{1}{\omega^2 - k^2}J^{\mu\lambda} - \frac{1}{2}J'^{\mu}_{\mu}\frac{1}{\omega^2 - k^2}J_\lambda{}^\lambda. \quad (5.3)$$

We shall evaluate the first term in the interaction amplitude; this gives

$$\begin{aligned}J'_{\mu\lambda}\frac{1}{\omega^2 - k^2}J^{\mu\lambda} &= \frac{1}{\omega^2 - k^2}\{J'_{00}J_{00} - J'_{01}J_{01} - J'_{02}J_{02} - J'_{03}J_{03} - J'_{10}J_{10}\\
&\quad - J'_{20}J_{20} - J'_{30}J_{30} + J'_{11}J_{11} + J'_{22}J_{22} + J'_{33}J_{33} + J'_{12}J_{12}\\
&\quad + J'_{21}J_{21} + J'_{13}J_{13} + J'_{31}J_{31} + J'_{23}J_{23} + J'_{32}J_{32}\}.\end{aligned}$$

Taking $k^\mu = (\omega, 0, 0, k)$ and using the conservation of the current (2.7) expressed in the momentum space

$$k^\mu J_{\mu\lambda} = 0, \qquad \omega J_{0\lambda} = -kJ_{3\lambda},$$

we shall get

$$\frac{1}{\omega^2-k^2}\left[\left(1-\frac{\omega^2}{k^2}\right)J'_{00}J_{00}-\left(1-\frac{\omega^2}{k^2}\right)J'_{01}J_{01}-\left(1-\frac{\omega^2}{k^2}\right)J'_{02}J_{02}-\left(1-\frac{\omega^2}{k^2}\right)J'_{03}J_{03}\right.$$
$$\left.-J'_{10}J_{10}-J'_{20}J_{20}+J'_{11}J_{11}+J'_{22}J_{22}+J'_{12}J_{12}+J'_{21}J_{21}+J'_{13}J_{13}+J'_{23}J_{23}\right].$$

Now using the second conservation law (2.8) in the momentum space

$$k^\lambda J_{\mu\lambda}=0,\qquad \omega J_{\mu 0}=-kJ_{\mu 3},$$

we arrive at

$$\frac{1}{\omega^2-k^2}\left[\left(1-\frac{\omega^2}{k^2}\right)J'_{00}J_{00}-\left(1-\frac{\omega^2}{k^2}\right)J'_{01}J_{01}-\left(1-\frac{\omega^2}{k^2}\right)J'_{02}J_{02}\right.$$
$$\left.-\left(1-\frac{\omega^2}{k^2}\right)J'_{10}J_{10}-\left(1-\frac{\omega^2}{k^2}\right)J'_{20}J_{20}-\left(1-\frac{\omega^2}{k^2}\right)\left(\frac{\omega^2}{k^2}\right)J'_{00}J_{00}\right]$$
$$+\frac{1}{\omega^2-k^2}[J'_{11}J_{11}+J'_{22}J_{22}+J'_{12}J_{12}+J'_{21}J_{21}]$$

and, after simple algebra, at

$$-\frac{1}{k^2}\left[\left(1-\frac{\omega^2}{k^2}\right)J'_{00}J_{00}-J'_{01}J_{01}-J'_{02}J_{02}-J'_{10}J_{10}-J'_{20}J_{20}\right]$$
$$+\frac{1}{\omega^2-k^2}[J'_{11}J_{11}+J'_{22}J_{22}+J'_{12}J_{12}+J'_{21}J_{21}].$$

Evaluating the second term in the interaction amplitude (5.2) in the same manner as above, we shall finally get for the total amplitude

$$-\frac{1}{k^2}\left[\left(1-\frac{\omega^2}{k^2}\right)J'_{00}J_{00}-J'_{01}J_{01}-J'_{02}J_{02}-J'_{10}J_{10}-J'_{20}J_{20}\right]$$
$$+\frac{1}{\omega^2-k^2}\left[\frac{1}{2}(J'_{11}-J'_{22})(J_{11}-J_{22})+J'_{12}J_{12}+J'_{21}J_{21}\right]. \tag{5.4}$$

For the instantaneous term we get

$$-\frac{1}{k^2}\left[\left(1-\frac{\omega^2}{k^2}\right)J'_{00}J_{00}-J'_{01}J_{01}-J'_{02}J_{02}-J'_{10}J_{10}-J'_{20}J_{20}\right] \tag{5.5}$$

and for the retarded term ($J_{12}\neq J_{21}$)

$$\frac{1}{\omega^2-k^2}\left[\frac{1}{2}(J'_{11}-J'_{22})(J_{11}-J_{22})+J'_{12}J_{12}+J'_{21}J_{21}\right]. \tag{5.6}$$

The retarded term represents a sum of three independent products,

$$+\tfrac{1}{4}[J'_{11}-J'_{22}+\mathrm{i}(J'_{12}+J'_{21})][J_{11}-J_{22}-\mathrm{i}(J_{12}+J_{21})]$$
$$+\tfrac{1}{4}[J'_{11}-J'_{22}-\mathrm{i}(J'_{12}+J'_{21})][J_{11}-J_{22}+\mathrm{i}(J_{12}+J_{21})]$$
$$+\tfrac{1}{2}(J'_{12}-J'_{21})(J'_{12}-J'_{21}), \tag{5.7}$$

or polarizations corresponding to the helicities $\lambda=\pm 2,0$. Thus all negative-norm states are excluded from the spectrum of the second-rank tensor gauge field $A_{\mu\lambda}$, due to the gauge invariance of the theory and we come to the conclusion that the theory does indeed respect unitarity at the free level.

6. Conclusion

In this paper, we are constructing perturbation theory in the coupling constant g. All qualities: the Lagrangian (1.1) $\mathcal{L} = K + gM + g^2N$, the field strength tensors (1.3) $G = F + g[A, A]$, the Bianchi identities (A.4), the exact equation of motion (3.1) and exact extended gauge transformations δ and $\tilde{\delta}$ can be consistently expanded in powers of g. As a first step, we are considering the properties of the theory in zero order of g and then its interactions. The above consideration comprises the complete analysis of the spectrum for the second-rank tensor gauge field and its tree-level interactions.

The Lagrangian (1.1) contains tensor gauge fields of all ranks. We have here an example of field theory with infinitely many interacting fields. Our knowledge of such field theories is limited. What we can do is to study their properties in steps. In particular, we have concentrated here on lower-rank tensor gauge field. More should be done in order to understand the structure of the particle spectrum at higher levels.

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Appendix A. Bianchi identity

The non-Abelian tensor fields $A^a_{\mu\lambda_1\cdots\lambda_s}$ can be seen as appearing in the expansion of the extended gauge field $\mathcal{A}_\mu(x, e)$ over the unit tangent vector e_λ [42–44],

$$\mathcal{A}_\mu(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} A^a_{\mu\lambda_1\cdots\lambda_s}(x) L^a e_{\lambda_1} \cdots e_{\lambda_s}.$$

and the extended field strength tensor can be defined in terms of the extended gauge field $\mathcal{A}_\mu(x, e)$ as follows:

$$\mathcal{G}_{\mu\nu}(x, e) = \partial_\mu \mathcal{A}_\nu(x, e) - \partial_\nu \mathcal{A}_\mu(x, e) - \mathrm{i}g[\mathcal{A}_\mu(x, e), \mathcal{A}_\nu(x, e)].$$

Defining the extended covariant derivative: $\mathcal{D}_\mu = \partial_\mu - \mathrm{i}g\mathcal{A}_\mu$, one can get [44]

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = [\partial_\mu - \mathrm{i}g\mathcal{A}_\mu, \partial_\nu - \mathrm{i}g\mathcal{A}_\nu] = -\mathrm{i}g\mathcal{G}_{\mu\nu}. \tag{A.1}$$

The operators $\mathcal{D}_\mu, \mathcal{D}_\nu, \mathcal{D}_\lambda$ obey Jacobi identity,

$$[\mathcal{D}_\mu, [\mathcal{D}_\nu, \mathcal{D}_\lambda]] + [\mathcal{D}_\nu, [\mathcal{D}_\lambda, \mathcal{D}_\mu]] + [\mathcal{D}_\lambda, [\mathcal{D}_\mu, \mathcal{D}_\nu]] = 0,$$

which with the aid of (A.1) is transformed into the generalized Bianchi identity

$$[\mathcal{D}_\mu, \mathcal{G}_{\nu\lambda}] + [\mathcal{D}_\nu, \mathcal{G}_{\lambda\mu}] + [\mathcal{D}_\lambda, \mathcal{G}_{\mu\nu}] = 0. \tag{A.2}$$

Let us now expand equation (A.2) over e_ρ up to the linear terms. We have

$$[\partial_\mu - \mathrm{i}gA_\mu - \mathrm{i}gA_{\mu\rho}e^\rho, G_{\nu\lambda} + G_{\nu\lambda,\rho}e^\rho] + \text{cyc.perm.} + O(e^2) = 0.$$

In zero order the above equation gives the standard Bianchi identity in YM theory,

$$[D_\mu, G_{\nu\lambda}] + [D_\nu, G_{\lambda\mu}] + [D_\lambda, G_{\mu\nu}] = 0, \tag{A.3}$$

where $D_\mu = \partial_\mu - \mathrm{i}gA_\mu$. The linear term in e_ρ gives

$$\begin{aligned}&[D_\mu, G_{\nu\lambda,\rho}] - \mathrm{i}g[A_{\mu\rho}, G_{\nu\lambda}] + [D_\nu, G_{\lambda\mu,\rho}] \\ &\quad -\mathrm{i}g[A_{\nu\rho}, G_{\lambda\mu}] + [D_\lambda, G_{\mu\nu,\rho}] - \mathrm{i}g[A_{\lambda\rho}, G_{\mu\nu}] = 0.\end{aligned} \tag{A.4}$$

Using explicit form of the operators D_μ, $G_{\mu\nu}$ and $G_{\mu\nu,\lambda}$ one can independently check the last identity and get convinced that it holds. Now, if we expand the above equation over g, the zeroth order gives the Bianchi identity for the free-field strength tensor $F_{\nu\lambda,\rho}$,

$$\partial_\mu F_{\nu\lambda,\rho} + \partial_\nu F_{\lambda\mu,\rho} + \partial_\lambda F_{\mu\nu,\rho} = 0. \tag{A.5}$$

These equations impose tight restrictions on the source currents and hence on the nature of interactions.

Appendix B.

The matrix operator in gravity (3.12) is of dimension 10×10 and has the following form:

$$\begin{matrix}
0 & 0 & 0 & 0 & -k^2 & 0 & 0 & -k^2 & 0 & 0 \\
0 & -\frac{k^2}{2} & 0 & 0 & 0 & 0 & -\frac{kw}{2} & 0 & 0 & 0 \\
0 & 0 & -\frac{k^2}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{kw}{2} & 0 \\
0 & 0 & 0 & 0 & kw & 0 & 0 & kw & 0 & 0 \\
-k^2 & 0 & 0 & -kw & 0 & 0 & 0 & k^2 - w^2 & 0 & -w^2 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(-k^2 + w^2) & 0 & 0 & 0 & 0 \\
0 & \frac{kw}{2} & 0 & 0 & 0 & 0 & \frac{w^2}{2} & 0 & 0 & 0 \\
-k^2 & 0 & 0 & -kw & k^2 - w^2 & 0 & 0 & 0 & 0 & -w^2 \\
0 & 0 & \frac{kw}{2} & 0 & 0 & 0 & 0 & 0 & \frac{w^2}{2} & 0 \\
0 & 0 & 0 & 0 & -w^2 & 0 & 0 & -w^2 & 0 & 0
\end{matrix} \tag{B.1}$$

and for the antisymmetric tensor field (3.16) it is of dimension 6×6,

$$\begin{matrix}
-\frac{k^2}{2} & 0 & 0 & 0 & \frac{kw}{2} & 0 \\
0 & -\frac{k^2}{2} & 0 & 0 & 0 & \frac{kw}{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(-k^2 + w^2) & 0 & 0 \\
-\frac{kw}{2} & 0 & 0 & 0 & \frac{w^2}{2} & 0 \\
0 & -\frac{kw}{2} & 0 & 0 & 0 & \frac{w^2}{2}.
\end{matrix} \tag{B.2}$$

The matrix operator for non-Abelian tensor gauge theory (3.4) is of dimension 16×16 and has the following explicit form:

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & -\frac{k^2}{2} & 0 & 0 & 0 & 0 & -\frac{k^2}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -k^2 & 0 & 0 & \frac{k^2}{2} & 0 & 0 & \frac{k\omega}{2} & 0 & 0 & 0 & 0 & 0 & -k\omega & 0 & 0 \\
0 & 0 & -k^2 & 0 & 0 & 0 & 0 & 0 & \frac{k^2}{2} & 0 & 0 & \frac{k\omega}{2} & 0 & 0 & -k\omega & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{k\omega}{2} & 0 & 0 & 0 & 0 & \frac{k\omega}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{k^2}{2} & 0 & 0 & -k^2 & 0 & 0 & -k\omega & 0 & 0 & 0 & 0 & 0 & \frac{k\omega}{2} & 0 & 0 \\
-\frac{k^2}{2} & 0 & 0 & -\frac{k\omega}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(k^2 - \omega^2) & 0 & -\frac{k\omega}{2} & 0 & 0 & -\frac{\omega^2}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & -k^2 + \omega^2 & 0 & 0 & \frac{1}{2}(k^2 - \omega^2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{k\omega}{2} & 0 & 0 & k\omega & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega^2}{2} & 0 & 0 \\
0 & 0 & \frac{k^2}{2} & 0 & 0 & 0 & 0 & 0 & -k^2 & 0 & 0 & -k\omega & 0 & 0 & \frac{k\omega}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(k^2 - \omega^2) & 0 & 0 & -k^2 + \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{k^2}{2} & 0 & 0 & -\frac{k\omega}{2} & 0 & \frac{1}{2}(k^2 - \omega^2) & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k\omega}{2} & 0 & 0 & -\frac{\omega^2}{2} \\
0 & 0 & -\frac{k\omega}{2} & 0 & 0 & 0 & 0 & 0 & k\omega & 0 & 0 & \omega^2 & 0 & 0 & -\frac{\omega^2}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{k\omega}{2} & 0 & 0 & 0 & 0 & \frac{k\omega}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & k\omega & 0 & 0 & -\frac{k\omega}{2} & 0 & 0 & -\frac{\omega^2}{2} & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\
0 & 0 & k\omega & 0 & 0 & 0 & 0 & 0 & -\frac{k\omega}{2} & 0 & 0 & -\frac{\omega^2}{2} & 0 & 0 & \omega^2 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\omega^2}{2} & 0 & 0 & 0 & 0 & -\frac{\omega^2}{2} & 0 & 0 & 0 & 0 & 0
\end{matrix} \tag{B.3}$$

and allows us to calculate its rank as a function of momenta.

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